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Yann Rébillé* Lionel Richefort†

Abstract

A directed network game of imperfect strategic substitutes with heterogeneous players is analyzed. We consider concave additive separable utility functions that encompass the quasi-linear ones. It is found that pure strategy Nash equilibria verify a non-linear complementarity problem. By requiring appropriate concavity in the utility functions, the existence of an equilibrium point is shown and equilibrium uniqueness is established with a P-matrix. Then, it appears that previous findings on network structure and sparsity hold for many more games.

Keywords: network game, additive preferences, complementarity problem, P-matrix.

JEL: C72, D85, L14.

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1 Introduction

It has been found that the linear complementarity problem (henceforth LCP) is a fundamental mathematical problem associated with major economic problems such as linear and quadratic programming problems, bimatrix game problems, as well as more general equilibrium problems (Cottle et al., 1992). In this present paper, we are interested in a new application of complementarity problem: finding a Nash equilibrium in network game problems. This class of games has the property that an action of a player affects the marginal payoff of his neighbors, i.e., a player's payoff depends on own effort as well as on the efforts exerted by his neighbors (Jackson, 2008).

Specifically, we concentrate on network games where efforts are strategic substitutes. That is, an increase in one player's strategy makes the best response of his neighbors decrease.[1] The literature in this area has focused on games where preferences are quasi-linear and Nash equilibria in pure strategies (henceforth PSNE) exist. The analysis of equilibrium uniqueness is then achieved by studying the following LCP (Corbo et al., 2007; Ballester et al., 2010). Given a network of relationships $\mathbf{\Lambda} \in \mathbb{R}_+^{N\times N}$ and a vector of peaks[2] $\mathbf{e}^* >> \mathbf{0}$, determine an effort profile $\hat{\mathbf{e}} \geq \mathbf{0}$ such that

$$(\mathbf{I}+\mathbf{\Lambda})^\mathsf{T}\,\hat{\mathbf{e}} - \mathbf{e}^* \geq \mathbf{0}, \quad \left[(\mathbf{I}+\mathbf{\Lambda})^\mathsf{T}\,\hat{\mathbf{e}} - \mathbf{e}^*\right]^\mathsf{T}\hat{\mathbf{e}} = \mathbf{0}, \tag{1}$$

where $\mathbf{I}$ stands for the identity matrix and the superscript T denotes the transpose of a matrix or a vector.[3] The first inequality checks that the sum between a player's own effort and the effort exerted by his predecessors is not less than his peak. The second equality states that a player exerts no effort if the total effort exerted by his predecessors is equal or higher than his peak.

In such games, sufficient conditions for the uniqueness of PSNE have been established with $\mathbf{I}+\mathbf{\Lambda}$ belonging to particular classes of matrices, and have then be related to some eigenvalues of $\mathbf{\Lambda}$: the largest eigenvalue (or spectral radius) (Ballester et al., 2006; Corbo et al., 2007; Ballester and

[1] This outcome may arise in local public good games, when the players in a neighborhood have an incentive to "free-ride" (Bramoullé and Kranton, 2007; Bloch and Zenginobuz, 2007; Corbo et al., 2007; Galeotti et al., 2010; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2011; Allouch, 2012), or in games of common property resources, when the players have an incentive to exert too much effort, i.e., to consume more than the efficient level (İlkiliç, 2010).

[2] In these games, it is assumed that each player l has a unique effort level e_l^* that maximizes his preferences. Thus, e_l^* is player l's peak. It corresponds to a "satiation" point above which marginal utility becomes negative.

[3] We keep this notation for the rest of the paper.

Calvó-Armengol, 2010) and the lowest eigenvalue (Bramoullé et al., 2011; Allouch, 2012). The eigenvalues of a network adjacency matrix carries details about local structural properties of networks (Cvetković et al., 1997). There are two main results. First, using the Perron-Frobenius theory of nonnegative matrices, it has been found that equilibrium uniqueness is guaranteed for all $\mathbf{e}^* >> \mathbf{0}$ whenever the spectral radius of $\mathbf{\Lambda}$ is low enough (Ballester et al., 2010). This condition derives from contraction of the best response functions and holds for directed and undirected networks. It says that there exists a unique equilibrium point whenever the density of the link structure is sufficiently low.

Second, using the theory of potential games and spectral graph theory, a sharper condition has been found: the LCP specified by (1) admits a unique solution for all $\mathbf{e}^* >> \mathbf{0}$ if the lowest eigenvalue of $\mathbf{\Lambda}$ is high enough (Bramoullé et al., 2011). This result provides the best known sufficient condition for the uniqueness of PSNE in these games (when the spectral radius is low enough, the lowest eigenvalue is high enough, but not vice versa). It entails that there exists a unique equilibrium point whenever the sparsity (or tightness) of the link structure is sufficiently high. It is, however, based on two important assumptions: (i) the network is undirected, i.e., $\mathbf{\Lambda}$ is symmetric, and (ii) best response functions are piece-wise linear. In fact, there has been few attempts to establish uniqueness when best response functions are non-linear, and all of them consider the case of undirected networks, a special case of directed networks (Bramoullé et al., 2011; Allouch, 2012).

This present paper tackles the problem of equilibrium uniqueness in directed network games of imperfect strategic substitutes with non-linear best response functions. The focus here is on concave additive separable utility functions that encompass the quasi-linear ones. First, we find that PSNE are characterized by a complementarity problem that extends the problem specified by (1). Second, by requiring appropriate concavity in the utility functions, the existence of a PSNE is shown. Then, using the methodology introduced by Rosen (1965) for concave games, we establish the uniqueness of PSNE with a P-matrix[4] (Fielder and Pták, 1962). As a corollary, it appears that previous findings related to network structure and sparsity (Bramoullé et al., 2011) hold for more general preferences, and therefore for many more games.

The remainder of the paper is organized as follows. Section 2 defines the game. Sections 3 and 4 establish equilibrium existence and equilibrium uniqueness, respectively. Section 5 discusses the relationship between uniqueness and network sparsity. Section 6 concludes. Note that all vectors con-

[4]A formal definition of P-matrix is given in Section 4.

sidered in this paper are column vectors, and are denoted by lowercase bold letters. We reserve the use of uppercase bold letters for matrices.

2 The model

There are N players and the set of players is $\mathcal{N} = \{1, \ldots, N\}$. Let e_l denotes player's l *individual effort* and $\mathbf{e} = (e_1, \ldots, e_N) \in \mathbb{R}_+^N$ an effort profile of all players. Players are arranged in a weighted and directed network. The basic representation of the network is given by its $N \times N$ weighted adjacency matrix $\mathbf{\Lambda}$, where $\lambda_{kl} \in [0, \infty)$ denotes the intensity of the link from player k to player l. We may interpret it as the rate of substitution from player k's effort to player l's effort (by convention, $\lambda_{kk} = 0$ for all k). Notice that the network is directed and weighted, i.e., $\mathbf{\Lambda}$ is asymmetric and nonnegative.

We shall consider from now on preferences that admit an additive separable utility representation with respect to *individual effort* and *collective effort*. The utility of an agent l is given by

$$u_l\left(e_l, E_l\right) = v_l\left(e_l\right) + w_l\left(E_l\right),$$

where player's l *collective effort* is given by $E_l = e_l + \sum_{k:k\neq l} \lambda_{kl} e_k$. Since collective effort is based on the collection of individual efforts, we may assume that the utility function is defined for all (e_l, E_l) such that $E_l, e_l \geq 0$. Utility is taken to be cardinal and may admit an interpretation in terms of benefits and costs (see Remark 1).

Technical assumptions

• for all l, $v_l'(0) + w_l'(0) > 0$,
• for all l, $v_l'(\infty) + w_l'(\infty) < 0$,
• for all l, v_l and w_l are continuously differentiable and concave, with v_l or w_l strictly concave.

The first assumption implies that under autarky (i.e., if the network is empty) the agent will provide some effort, otherwise in any network he will free-ride (we do not assume, however, that preferences are monotonic). The second assumption implies that the preferences are single-peaked w.r.t. individual effort for any given level of collective effort and allows a trade-off between individual and collective effort. The third assumption reflects the convexity of preferences.

We assume that the strategy of utility-maximizing individual effort involves a simultaneous-move game $\mathcal{G}(\mathbf{\Lambda}, \mathbf{v}, \mathbf{w})$, where $(\mathbf{v}, \mathbf{w})$ denote the preferences profile of the players. For all players, utility maximization occurs where the effort level is such that marginal utility is equal to zero. Letting $\widetilde{\mathbf{\Lambda}} = \mathbf{I} + \mathbf{\Lambda}$, we find that all PSNE admitted by $\mathcal{G}$ are characterized by a non-linear complementarity problem.

Property 1. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{v}, \mathbf{w})$ be a network game. Then, a profile $\hat{\mathbf{e}} \in \mathbb{R}_+^N$ is a* PSNE *if and only if $\hat{\mathbf{e}}$ satisfies*

$$\mathbf{v}'(\hat{\mathbf{e}}) + \mathbf{w}'\left(\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}\hat{\mathbf{e}}\right) \leq \mathbf{0}, \quad \left[\mathbf{v}'(\hat{\mathbf{e}}) + \mathbf{w}'\left(\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}\hat{\mathbf{e}}\right)\right]^{\mathsf{T}} \hat{\mathbf{e}} = \mathbf{0}, \tag{2}$$

where for all l, $(\mathbf{v}'(\hat{\mathbf{e}}))_l = v_l'(\hat{e}_l)$ and $(\mathbf{w}'(\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}\hat{\mathbf{e}}))_l = w_l'(\hat{e}_l + \sum_{k:k\neq l} \lambda_{kl}\hat{e}_k)$.

The proof is straightforward from the Karush-Kuhn-Tucker's conditions of utility maximization with individual effort constrained to be nonnegative. Then, we observe that the complementarity problem specified by (2) extends the LCP specified by (1).

Remark 1 (Public goods in networks). To fix ideas on the generality of the problem analyzed in this paper, consider a network game with quasi-linear preferences, i.e., for all l,

$$u_l(e_l, E_l) = v_l e_l + w_l(E_l),$$

where v_l is a constant, $w_l'(0) > -v_l > w_l'(\infty)$ and $w_l'' < 0$. In this context, a profile $\hat{\mathbf{e}} \in \mathbb{R}_+^N$ is a PSNE if and only if $\hat{\mathbf{e}}$ satisfies

$$\mathbf{v} + \mathbf{w}'\left(\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}\hat{\mathbf{e}}\right) \leq \mathbf{0}, \quad \left[\mathbf{v} + \mathbf{w}'\left(\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}\hat{\mathbf{e}}\right)\right]^{\mathsf{T}} \hat{\mathbf{e}} = \mathbf{0}.$$

If, for all l, $-v_l$ is in the range of w_l', the problem leads to a LCP where $\hat{\mathbf{e}} \in \mathbb{R}_+^N$ is such that

$$\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}\hat{\mathbf{e}} - (\mathbf{w}')^{-1}(-\mathbf{v}) \geq \mathbf{0}, \quad \left[\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}\hat{\mathbf{e}} - (\mathbf{w}')^{-1}(-\mathbf{v})\right]^{\mathsf{T}} \hat{\mathbf{e}} = \mathbf{0},$$

which turns out to be equivalent to the LCP specified by (1), since $\mathbf{e}^* >> \mathbf{0}$ is the vector such that, for all l, $v_l + w_l'(e_l^*) = 0$. This model, pioneered by Bramoullé and Kranton (2007)[5], is the traditional model analyzed in the literature on local public goods games (Bloch and Zenginobuz, 2007; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2011). It is therefore a limiting case of the model developed and analyzed in this present paper.

The next section presents sufficient conditions pertaining the existence of a solution to the problem specified by (2).

[5]In Bramoullé and Kranton (2007), the utility of an agent l is given by

$$u_l(e_l, E_l) = b_l(E_l) - c_l e_l,$$

where b_l is a twice-differentiable strictly concave benefit function with $b_l(0) = 0$, $b_l' > 0$ and $b_l'' < 0$, and $c_l < b_l'(0)$ is a constant that denotes the individual marginal cost of effort.

3 Equilibrium existence

In games where PSNE are characterized by the LCP specified by (1), the proof of equilibrium existence is straightforward: the set of individual best responses defines a piece-wise linear mapping from a convex compact subset of a Euclidean space into itself, and the existence of an equilibrium point is guaranteed. Establishing the existence of a solution to the complementarity problem specified by (2) is less immediate. To achieve this task, we follow an analytic approach to check that the best response of each agent satisfies the hypothesis of Brouwer's Fix Point Theorem. We obtain the following result.

Theorem 1. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{v}, \mathbf{w})$ be a network game. Then, $\mathcal{G}$ admits a PSNE whenever each agent satisfies the technical conditions.*

Proof. Let l be an agent. Since $u_l = v_l + w_l$ is stricly concave, $u_l'(0) = v_l'(0) + w_l'(0) > 0$ and $u_l'(M) = v_l'(M) + w_l'(M) < 0$ for some $M > 0$ there exists a unique $e_l^* > 0$ such that $u_l'(e_l^*) = v_l'(e_l^*) + w_l'(e_l^*) = 0$. Moreover, e_l^* is l's maximum.

Let $F_l \geq 0$ be a level of collective effort exerted by other agents than l. Agent's l utility is given by

$$u_l(e_l, e_l + F_l) = v_l(e_l) + w_l(e_l + F_l).$$

By assumption, $u_l(.,. + F_l)$ is strictly concave, so $u_l(.,. + F_l)'$ is strictly decreasing and continuous. The best response given F_l is

$$b_l(F_l) = \begin{cases} [u_l(.,. + F_l)']^{-1}(0) & , \text{if } u_l(0, 0 + F_l)' > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Since

$$u_l(.,. + F_l)' \leq u_l(.,.)'$$

we have

$$b_l(F_l) \leq b_l(0) = e_l^*.$$

So the autarkic equilibrium effort is always greater than the equilibrium effort in a network.

Let us check that the best response is continuous w.r.t. F_l. Let $F_l \geq 0$.

<u>1^{st} case:</u> $u_l(0, 0 + F_l)' = v_l'(0) + w_l'(F_l) < 0$.
Then, $b_l(F_l) = 0$. Since w_l' is continous, there exists some neighborhood V of F_l such that $v_l'(0) + w_l'(F) < 0$ for $F \in V$. Thus, $b_l(F) = 0$ for $F \in V$ so b_l is continuous at F_l.

$\underline{2^{nd}\text{ case:}}$ $u_l\left(0,0+F_l\right)'=v_l'\left(0\right)+w_l'\left(F_l\right)\geq 0$.
By definition, $\left(e_l,F_l\right)$ with $e_l=b_l\left(F_l\right)$ is a solution to the equation

$$x_l\left(e,F\right)=v_l'\left(e\right)+w_l'\left(e+F\right)=0.$$

Now, $\frac{\partial x_l}{\partial e}\left(e_l,F_l\right)=v_l''\left(e_l\right)+w_l''\left(e_l+F_l\right)<0$ by strict-concavity, so according to the implicite function theorem there exists some differentiable and invertible function γ such that

$$F_l=\gamma\left(e_l\right)$$

on some open neighbourhood V of e_l, satisfying

$$x_l\left(e,\gamma\left(e\right)\right)=v_l'\left(e\right)+w_l'\left(e+\gamma\left(e\right)\right)=0.$$

Thus, $b_l\left(F_l\right)=e_l=\gamma^{-1}\left(F_l\right)$ on $U=\gamma^{-1}\left(V\right)\ni F_l$, so b_l is continuous at F_l.
Consider the mapping

$$\begin{array}{rccc} B: & \prod_l\left[0,e_l^*\right] & \longrightarrow & \prod_l\left[0,e_l^*\right] \\ & \mathbf{e} & \mapsto & \left(b_l\left(\sum_{k:k\neq l}\lambda_{kl}e_k\right)\right)_l \end{array}$$

B is continuous w.r.t. $\mathbf{e}$ since b_l and $\left(\mathbf{e}\mapsto\sum_{k:k\neq l}\lambda_{kl}e_k\right)$ are continuous for all l. According to Brouwer's Fix Point Theorem, B admits a fix point $\mathbf{e}$ and $\mathbf{e}$ is a PSNE of the network game $\mathcal{G}$ by construction. $\square$

The next section is devoted to the special case of a unique PSNE to the game $\mathcal{G}$. Following Cottle et al. (1992)'s terminology, we present a *global* uniqueness result[6] as we provide conditions under which the solvable complementarity problem specified by (2) has only one solution. From now on, we assume that each agent satisfies the technical conditions.

4 Equilibrium uniqueness

It is well-known that matrix classes play a central role in the theory of the LCP. One of the most fundamental class is the one consisting of P-matrices.[7]

[6] By contrast, Bramoullé et al. (2011) present a *local* uniqueness result since their proof, based on Taylor approximation of best responses, provides conditions under which a given equilibrium is the only equilibrium in one of its neighborhoods.

[7] Cottle et al. (1992) defines the traditional LCP as follows. Given $\mathbf{M}\in\mathbb{R}^{N\times N}$ and $\mathbf{q}\in\mathbb{R}^N$, determine $\mathbf{z}\geq\mathbf{0}$ such that

$$\mathbf{Mz}+\mathbf{q}\geq\mathbf{0},\quad\left(\mathbf{Mz}+\mathbf{q}\right)^{\mathsf{T}}\mathbf{z}=\mathbf{0}.$$

It is well-known that the above LCP has a unique solution for all vectors $\mathbf{q}$ if and only if $\mathbf{M}$ is a P-matrix (Samelson et al., 1958). See also Theorem 3.3.7 in Cottle et al. (1992, p.148).

In this section, we establish the uniqueness of solutions to the complementarity problem specified by (2) with such a matrix. Let us introduce the class of P-matrices.

Definition 1 (Fiedler and Pták, 1962)**.** Let $\mathbf{M}$ be a real matrix. Then, $\mathbf{M}$ is said to be a *P-matrix* if all its principal minors are positive.

An essential feature of P-matrix is that it does not reverse the sign of any nonzero vector (Fiedler and Pták, 1962), i.e.,

$$\forall\ \mathbf{x} \neq \mathbf{0},\ \exists\ k\ /\ x_k \left(\mathbf{M}\mathbf{x}\right)_k > 0.$$

Note that $\mathbf{M}^{\mathsf{T}}$ is a P-matrix whenever $\mathbf{M}$ is a P-matrix.[8]

To establish equilibrium uniqueness, we follow the analytic approach introduced by Rosen (1965) for proving uniqueness in concave games. We obtain the following result.

Theorem 2. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{v}, \mathbf{w})$ be a network game. Then, $\mathcal{G}$ admits a unique PSNE whenever $\widetilde{\mathbf{\Lambda}}$ is a P-matrix.*

Proof. Let l be an agent. Each agent l satisfies its maximisation program given others efforts, $\left(e_k\right)_{k:k\neq l}$,

$$\max\ v_l\left(e_l\right) + w_l\left(e_l + F_l\right):\ \text{s.t. } e_l \geq 0$$

where $F_l = \sum_{k:k\neq l} \lambda_{kl} e_k$.

Let μ_l be the Karush-Kuhn-Tucker's multiplier associated to the constraint $-e_l \leq 0$. Under the technical conditions, $v_l\left(.\right) + w_l\left(. + F_l\right)$ is strictly concave and differentiable. The first order conditions are therefore sufficient,

$$v_l'\left(e_l\right) + w_l'\left(e_l + F_l\right) + \mu_l = 0$$

with

$$\mu_l e_l = 0,\ \mu_l \geq 0.$$

Let us assume that there are two PSNE, $\mathbf{e}^0 \neq \mathbf{e}^1$. Let $\{r_l\}_l \in \mathbb{R}_+^N$. Multiplying agent's l first order condition at $\mathbf{e}^0$ by $r_l\left(e_l^1 - e_l^0\right)$ and agent's l first order condition at $\mathbf{e}^1$ by $r_l\left(e_l^0 - e_l^1\right)$ we have,

$$r_l\left(e_l^1 - e_l^0\right)\left[v_l'\left(e_l^0\right) + w_l'\left(e_l^0 + \sum_{k:k\neq l} \lambda_{kl} e_k^0\right)\right] + r_l\left(e_l^1 - e_l^0\right)\mu_l^0 = 0,$$

[8] See Plemmons (1977) for a survey of the properties associated to real square matrices whose principal minors are positive.

$$r_l\left(e_l^0-e_l^1\right)\left[v_l'\left(e_l^1\right)+w_l'\left(e_l^1+\sum_{k:k\neq l}\lambda_{kl}e_k^1\right)\right]+r_l\left(e_l^0-e_l^1\right)\mu_l^1=0.$$

Adding up both equalities and summing over l, it comes

$$\begin{aligned}
&\textstyle\sum_l r_l\left(e_l^1-e_l^0\right)\left[v_l'\left(e_l^0\right)+w_l'\left(e_l^0+\sum_{k:k\neq l}\lambda_{kl}e_k^0\right)\right]\\
+\ &\textstyle\sum_l r_l\left(e_l^0-e_l^1\right)\left[v_l'\left(e_l^1\right)+w_l'\left(e_l^1+\sum_{k:k\neq l}\lambda_{kl}e_k^1\right)\right]\\
+\ &\textstyle\sum_l r_l\left(e_l^1-e_l^0\right)\mu_l^0+\sum_l r_l\left(e_l^0-e_l^1\right)\mu_l^1\\
&=0.
\end{aligned}$$

Now, for all l, $\mu_l^0e_l^0=0$ and $\mu_l^1e_l^1=0$, thus,

$$\begin{aligned}
&\textstyle\sum_l r_l\left(e_l^1-e_l^0\right)\left[v_l'\left(e_l^0\right)+w_l'\left(e_l^0+\sum_{k:k\neq l}\lambda_{kl}e_k^0\right)\right]\\
+\ &\textstyle\sum_l r_l\left(e_l^0-e_l^1\right)\left[v_l'\left(e_l^1\right)+w_l'\left(e_l^1+\sum_{k:k\neq l}\lambda_{kl}e_k^1\right)\right]\\
+\ &\textstyle\sum_l r_le_l^1\mu_l^0+\sum_l r_le_l^0\mu_l^1\\
&=0.
\end{aligned}$$

Since $e_l^0,e_l^1,\mu_l^0,\mu_l^1,r_l\geq 0$ it holds

$$\begin{aligned}
&\textstyle\sum_l r_l\left(e_l^1-e_l^0\right)\left[v_l'\left(e_l^0\right)+w_l'\left(e_l^0+\sum_{k:k\neq l}\lambda_{kl}e_k^0\right)\right]\\
+\ &\textstyle\sum_l r_l\left(e_l^0-e_l^1\right)\left[v_l'\left(e_l^1\right)+w_l'\left(e_l^1+\sum_{k:k\neq l}\lambda_{kl}e_k^1\right)\right]\\
&\leq 0
\end{aligned}$$

or

$$\begin{aligned}
Ineq._r:\ \sum_l r_l\left(e_l^1-e_l^0\right)&\left[v_l'\left(e_l^0\right)+w_l'\left(e_l^0+\sum_{k:k\neq l}\lambda_{kl}e_k^0\right)\right.\\
&\left.-\left(v_l'\left(e_l^1\right)+w_l'\left(e_l^1+\sum_{k:k\neq l}\lambda_{kl}e_k^1\right)\right)\right]\leq 0.
\end{aligned}$$

Let $\mathbf{x}=\mathbf{e}^1-\mathbf{e}^0\neq\mathbf{0}$. Since, $\widetilde{\mathbf{\Lambda}}$ is a P-matrix, $\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}$ is also a P-matrix. Then for some l_0,

$$x_{l_0}\left(\widetilde{\mathbf{\Lambda}}^{\mathsf{T}}\mathbf{x}\right)_{l_0}>0$$

that is,

$$\left(e^1_{l_0} - e^0_{l_0}\right)\left[\left(e^1_{l_0} + \sum_{k:k\neq l_0} \lambda_{kl_0} e^1_k\right) - \left(e^0_{l_0} + \sum_{k:k\neq l_0} \lambda_{kl_0} e^0_k\right)\right] > 0,$$

the two factors having same sign. Since v_{l_0} is concave, so v'_{l_0} is non-increasing, thus

$$\left(e^1_{l_0} - e^0_{l_0}\right)\left[v'_{l_0}\left(e^0_{l_0}\right) - v'_{l_0}\left(e^1_{l_0}\right)\right] \geq 0$$

and w_{l_0} is concave, so w'_{l_0} is non-increasing, thus

$$\left(e^1_{l_0} - e^0_{l_0}\right)\left[w'_{l_0}\left(e^0_{l_0} + \sum_{k:k\neq l_0} \lambda_{kl_0} e^0_k\right) - w'_{l_0}\left(e^1_{l_0} + \sum_{k:k\neq l_0} \lambda_{kl_0} e^1_k\right)\right] \geq 0,$$

with a strict inequality where strict concavity holds, so

$$\begin{aligned}\left(e^1_{l_0} - e^0_{l_0}\right)\Bigg[&\left(v'_{l_0}\left(e^0_{l_0}\right) + w'_{l_0}\left(e^0_{l_0} + \sum_{k:k\neq l_0} \lambda_{kl_0} e^0_k\right)\right)\\ &- \left(v'_{l_0}\left(e^1_{l_0}\right) + w'_{l_0}\left(e^1_{l_0} + \sum_{k:k\neq l_0} \lambda_{kl_0} e^1_k\right)\right)\Bigg] > 0.\end{aligned}$$

Choosing, $r_{l_0} = 1$ and $r_l = 0$ for $l \neq l_0$, we obtain from the $Ineq._r$

$$\begin{aligned}\left(e^1_{l_0} - e^0_{l_0}\right)\Bigg[&\left(v'_{l_0}\left(e^0_{l_0}\right) + w'_{l_0}\left(e^0_{l_0} + \sum_{k:k\neq l_0} \lambda_{kl_0} e^0_k\right)\right)\\ &- \left(v'_{l_0}\left(e^1_{l_0}\right) + w'_{l_0}\left(e^1_{l_0} + \sum_{k:k\neq l_0} \lambda_{kl_0} e^1_k\right)\right)\Bigg] \leq 0,\end{aligned}$$

a contradiction. So, $\mathbf{x} = \mathbf{e}^1 - \mathbf{e}^0 = \mathbf{0}$ and uniqueness is established. □

It is worthy to note that our uniqueness result carries over one change in the technical assumptions.

Remark 2 (Constrained effort). The second assumption in the technical assumptions, i.e., $v'(\infty) + w'(\infty) < 0$, implies that the agents' best responses take values in a bounded set. This assumption can be replaced directly by assuming that $e_l \in [0, M_l]$, for all $l \in \mathcal{N}$, where $0 \leq M_l < \infty$, the proof of equilibrium uniqueness being slightly modified to take account of the boundedness constraints.

5 Uniqueness and network sparsity

When best response functions are piece-wise linear, the uniqueness of PSNE is traditionally related to the largest and the lowest eigenvalues of the network adjacency matrix in order to obtain informations on the typical link structures that yield a unique equilibrium (Ballester et al., 2006; Ballester and Calvó-Armengol, 2010; Bramoullé et al., 2011; Allouch, 2012).[9]

When the network is directed, the adjacency matrix is asymmetric and its eigenvalues are generally complex numbers. By Theorem (3,3) of Fiedler and Pták (1962), the uniqueness condition we established with a P-matrix may be related to the lowest real eigenvalue of the network adjacency matrix and its principal submatrices. That is, the game $\mathcal{G}$ admits a unique PSNE if the lowest real eigenvalue of $\mathbf{\Lambda}$ and its principal submatrices is high enough. To our knowledge, however, there is no result that establishes a relationship between structural properties of directed networks and the lowest real eigenvalue of their associated adjacency matrix. Previous works on the spectra of directed network adjacency matrices has focused on the spectral radius of adjacency matrices (Brualdi, 2010).

When the network is undirected or acyclic, things go much better. We provide two results related to these particular link structures. The first one concerns undirected networks and extends Bramoullé et al. (2011)'s result to the case of additive separable preferences. A network is undirected if and only if its adjacency matrix is symmetric. Since real symmetric matrices are Hermitian, they do not admit complex eigenvalues. Let $\mu_{min}(\mathbf{\Lambda})$ be the lowest eigenvalue of the network adjacency matrix. We obtain the following result.

Corollary 1. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{v}, \mathbf{w})$ be a network game and let the network be undirected. Then, $\mathcal{G}$ admits a unique PSNE whenever $\mu_{min}(\mathbf{\Lambda}) > -1$.*

Proof. Let $\mathbf{\Lambda}$ be an undirected network adjacency matrix. Therefore, $\mathbf{\Lambda}$ is symmetric, and $\widetilde{\mathbf{\Lambda}}$ is also symmetric. Hence, its eigenvalues are real. Since $\mu_{min}(\mathbf{\Lambda}) > -1$ if and only if $\mu_{min}(\widetilde{\mathbf{\Lambda}}) > 0$, $\widetilde{\mathbf{\Lambda}}$ is a P-matrix. So, by Theorem 2, $\mathcal{G}$ admits a unique PSNE. □

Remark 3 (Sparsity of undirected networks). The lowest eigenvalue of a symmetric and nonnegative adjacency matrix is a standard measure of the sparsity of the network (see, e.g., Bell et al., 2008). The higher the lowest eigen-

[9]There is a large literature in algebraic graph theory that studies the properties of the eigenvalues and eigenvectors of adjacency matrices associated with a graph. One fundamental result entails that any graph is completely determined by its eigenvalues and eigenvectors. See Chapter 2 in Cvetović et al. (1997).

value, the sparser is the network. Then, Corollary 1 entails that the game $\mathcal{G}$ admits a unique PSNE whenever the network is sufficiently sparse.

The next result concerns acyclic networks. A typical example is a river network where links represent flows of water (Ambec and Sprumont, 2002). It is well-known that a triangular matrix is a P-matrix if and only if all its diagonal entries are strictly positive.[10] Since a network adjacency matrix is similar to a triangular matrix if and only if the network is acyclic, we obtain the following result.

Corollary 2. *Let $\mathcal{G}(\mathbf{\Lambda}, \mathbf{v}, \mathbf{w})$ be a network game and let the network be acyclic. Then, $\mathcal{G}$ admits a unique PSNE.*

Proof. Let $\mathbf{\Lambda}$ be an acyclic network adjacency matrix. Therefore, $\mathbf{\Lambda}$ is similar to a triangular matrix with diagonal entries equal to zero and all of its eigenvalues are zero. Thus, all the eigenvalues of $\widetilde{\mathbf{\Lambda}}$ are equal to 1, and consequently, its determinant is equal to 1. The same observation holds for all principal submatrices of $\widetilde{\mathbf{\Lambda}}$. It follows that $\widetilde{\mathbf{\Lambda}}$ is a P-matrix. Hence, by Theorem 2, $\mathcal{G}$ admits a unique PSNE. □

Remark 4 (Sparsity of acyclic networks). Corollary 2 entails that acyclic networks are always sufficiently sparse to guarantee uniqueness of PSNE to the game $\mathcal{G}$.

6 Conclusion

In this paper, we have explored a wide class of network games with additive separable preferences that yield non-linear best response functions. This class of games encompasses various well-known games including the voluntary contribution of public goods (Bramoullé and Kranton, 2007; Bloch and Zenginobuz, 2007; Bramoullé et al., 2011).

We discuss two possible extensions of this work. First, our analysis is restricted to additive utility functions. First-order conditions then produce a complementarity problem which is non-linear. Consider now general $\mathcal{C}^2$ utility functions $u_l\left(e_l, E_l\right)$ for all l that induce the following complementarity problem. Given a network of relationships $\mathbf{\Lambda} \in \mathbb{R}_+^{N\times N}$ and u_l twice differentiable and continuous functions for all l, determine an effort profile $\hat{\mathbf{e}} \geq \mathbf{0}$ such that

$$\mathbf{u}'\left(\hat{\mathbf{e}}, \widetilde{\mathbf{\Lambda}}^\mathsf{T}\hat{\mathbf{e}}\right) \leq \mathbf{0}, \quad \left[\mathbf{u}'\left(\hat{\mathbf{e}}, \widetilde{\mathbf{\Lambda}}^\mathsf{T}\hat{\mathbf{e}}\right)\right]^\mathsf{T} \hat{\mathbf{e}} = \mathbf{0},$$

[10] This is due to the fact that every principal minor of a triangular matrix equals a product of diagonal elements.

where for all l,

$$\left(\mathbf{u}'\left(\hat{\mathbf{e}},\widetilde{\boldsymbol{\Lambda}}^{\mathsf{T}}\hat{\mathbf{e}}\right)\right)_l=\frac{\partial u_l}{\partial e_l}\left(\hat{e}_l,\hat{e}_l+\sum_{k:k\neq l}\lambda_{kl}\hat{e}_k\right)+\frac{\partial u_l}{\partial E_l}\left(\hat{e}_l,\hat{e}_l+\sum_{k:k\neq l}\lambda_{kl}\hat{e}_k\right).$$

Establishing the existence of a unique solution to the above problem might require to find the appropriate shape in the utility functions. It is however a non-trivial issue.

Second, a characterization of the unique Nash equilibrium should be possible, at least when the equilibrium is interior. Letting $F_l \geq 0$ be a level of collective effort exerted by other agents than l, the utility of an agent l is given by $u_l\left(e_l, e_l+F_l\right)=v_l\left(e_l\right)+w_l\left(e_l+F_l\right)$. Assume there is a unique interior equilibrium $\hat{\mathbf{e}}$. It is such that, for all l, $v_l'\left(\hat{e}_l\right)+w_l'\left(\hat{e}_l+F_l\right)=0$. That is, when the range of $-v_l'$ is included in the range of w_l', $\hat{e}_l$ satisfies

$$F_l=f_l\left(\hat{e}_l\right)-\hat{e}_l$$

where $f_l=\left(w_l'\right)^{-1}\circ(-v_l')$. When best responses are piece-wise linear, $f_l\left(\hat{e}_l\right)=e_l^*$ a constant, and the unique equilibrium profile may be expressed in terms of a network centrality vector (Ballester and Calvó-Armengol, 2010; İlkiliç, 2010). Then, it would be pertinent to derive a structural solution to the above system of equalities. This is another open issue.

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